

SELF-APPROXIMATION OF DIRICHLET L -FUNCTIONS

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ABSTRACT. Let d be a real number, let s be in a fixed compact set of the strip $1/2 < \sigma < 1$, and let $L(s, \chi)$ be the Dirichlet L -function. The hypothesis is that for any real number d there exist ‘many’ real numbers τ such that the shifts $L(s+i\tau, \chi)$ and $L(s+id\tau, \chi)$ are ‘near’ each other. If d is an algebraic irrational number then this was obtained by T. Nakamura. L. Pańkowski solved the case then d is a transcendental number. We prove the case then $d \neq 0$ is a rational number. If $d = 0$ then by B. Bagchi we know that the above hypothesis is equivalent to the Riemann hypothesis for the given Dirichlet L -function. We also consider a more general version of the above problem.

1. INTRODUCTION

Let, as usual, $s = \sigma + it$ denote a complex variable. For $\sigma > 1$, the Dirichlet L -function is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi(n)$ is a Dirichlet character mod q . For $q = 1$ we get $L(s, \chi) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function.

In [6] Bohr proved that if χ is a nonprincipal character, then the Riemann hypothesis for $L(s, \chi)$ is equivalent to the almost periodicity of $L(s, \chi)$ in the half plane $\sigma > 1/2$. A function $f(s)$ is *almost periodic* in a region $E \subset \mathbb{C}$ if for any positive ε and any compact subset K in E there exists a sequence of real numbers $\dots < \tau_{-1} < 0 < \tau_1 < \tau_2 < \dots$ such that

$$\liminf_{m \rightarrow \pm\infty} (\tau_{m+1} - \tau_m) > 0, \quad \limsup_{m \rightarrow \pm\infty} \frac{\tau_m}{m} < \infty$$

and

$$|f(s + i\tau_m) - f(s)| < \varepsilon \quad \text{for all } s \in K \text{ and } m \in \mathbb{Z}$$

hold. Bohr [6] also obtained that every Dirichlet series is almost-periodic in its half-plane of absolute convergence. Effective upper bounds for the almost periodicity of Dirichlet series with Euler products in the half-plane of absolute convergence were considered by Girondo and Steuding [8]. Note that every Dirichlet L -function is almost periodic in the sense of Besicovitch on any vertical line of the strip $1/2 < \sigma < 1$. For this and related results see Besicovitch [5] and Maucclair [13], [14].

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Bagchi [2] proved that the Riemann hypothesis for $L(s, \chi)$ (χ is an arbitrary Dirichlet character) is true if and only if for any compact subset \mathcal{K} of the strip $1/2 < \sigma < 1$ and for any $\varepsilon > 0$

$$(1) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{s \in \mathcal{K}} |L(s + i\tau, \chi) - L(s, \chi)| < \varepsilon \right\} > 0,$$

where $\text{meas } A$ stands for the Lebesgue measure of a measurable set A . Bagchi says that the Dirichlet L -function $L(s, \chi)$ is *strongly recurrent* on the strip $\sigma_0 < \sigma < \sigma_1$ if (1) is valid for any compact \mathcal{K} of the strip $\sigma_0 < \sigma < \sigma_1$. The strong recurrence is connected with the universality property of Dirichlet series. More about the universality and the strong recurrence see Bagchi [1], [2], [3], and Steuding [17].

There are several unconditional results concerning the self-approximation of Dirichlet L -functions in the critical strip. Let \mathcal{K} be a compact subset of the strip $1/2 < \sigma < 1$ and let $\lambda \in \mathbb{R}$ be such that \mathcal{K} and $\mathcal{K} + i\lambda := \{s + i\lambda : s \in \mathcal{K}\}$ are disjoint. From Kaczorowski, Laurinćikas and Steuding [10] it follows that for any character χ and any $\varepsilon > 0$

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{s \in \mathcal{K}} |L(s + i\lambda + i\tau, \chi) - L(s + i\tau, \chi)| < \varepsilon \right\} > 0.$$

Nakamura [15] considered the joint universality of shifted Dirichlet L -functions. His Theorem 1.1 leads to the following statement. If $1 = d_1, d_2, \dots, d_m$ are algebraic real numbers *linearly independent* over \mathbb{Q} , then for any Dirichlet character χ and any $\varepsilon > 0$

$$(2) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j, k \leq m} \max_{s \in \mathcal{K}} |L(s + id_j\tau, \chi) - L(s + id_k\tau, \chi)| < \varepsilon \right\} > 0.$$

If $m = 2$ then Pańkowski [16] using Six Exponentials Theorem showed that (2) holds for d_1, d_2 are real numbers linearly independent over \mathbb{Q} .

We prove the following theorem.

Theorem 1. *Let $1 = d_1, d_2, \dots, d_m$ be nonzero algebraic real numbers and let \mathcal{K} be a compact subset of the strip $1/2 < \sigma < 1$. Then for any Dirichlet character χ and any $\varepsilon > 0$ the inequality (2) is valid.*

Note that Theorem 1 remains true if d_1, d_2, \dots, d_m are replaced by dd_1, dd_2, \dots, dd_m , where $d \in \mathbb{R}$. The next theorem shows that ‘lim inf’ in the inequality (2) often can be replaced by ‘lim’.

Theorem 2. *Let d_1, d_2, \dots, d_m be any real numbers, let $\chi_1, \chi_2, \dots, \chi_m$ be any Dirichlet characters, and let \mathcal{K} be a compact subset of the strip $1/2 < \sigma < 1$. Then for any $\varepsilon > 0$, except an at most countable set of ε , there exists a limit*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j, k \leq m} \max_{s \in \mathcal{K}} |L(s + id_j\tau, \chi_j) - L(s + id_k\tau, \chi_k)| < \varepsilon \right\}.$$

The mentioned results of Nakamura and Pańkowski together with Theorem 1 and Theorem 2 lead to the following corollary.

Corollary 3. *Let d be a nonzero real number and let \mathcal{K} be a compact subset of the strip $1/2 < \sigma < 1$. Then for any Dirichlet character χ and any $\varepsilon > 0$, except an at most countable set of ε ,*

$$(3) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{s \in \mathcal{K}} |L(s + i\tau, \chi) - L(s + id\tau, \chi)| < \varepsilon \right\} > 0.$$

From the proof of Theorem 2 we see that for any real numbers d_1, \dots, d_m and for any Dirichlet characters χ_1, \dots, χ_m the function

$$g(\tau) = \max_{1 \leq j, k \leq m} \max_{s \in \mathcal{K}} |L(s + id_j\tau, \chi_j) - L(s + id_k\tau, \chi_k)|$$

is Besicovitch almost periodic function (for the definition see Section 3 above the proof of Theorem 2). Let $\varepsilon > 0$ be such that the limit (3) exists. For such ε we define a characteristic function $I_\varepsilon(\tau)$, $\tau \in \mathbb{R}$, by

$$(4) \quad I_\varepsilon(\tau) = \begin{cases} 1, & \text{if } g(\tau) < \varepsilon, \\ 0, & \text{if } g(\tau) \geq \varepsilon. \end{cases}$$

It is known (Jessen and A. Wintner [7, Section 12]) that $I_\varepsilon(\tau)$ is Besicovitch almost periodic function also. Thus we can say that self-approximations of Dirichlet L -functions, considered in this paper, usually appear in a regular way.

Theorem 1 and Theorem 2 are proved in Section 3. Next we state several lemmas.

2. LEMMAS

We start from the following statement.

Lemma 4. *Let \mathcal{K} be a compact subset of the rectangle U . Let*

$$d = \min_{z \in \partial U} \min_{s \in \mathcal{K}} |s - z|.$$

If $f(s)$ is analytic on U and

$$\int_U |f(s)|^2 d\sigma dt \leq \varepsilon,$$

then

$$\max_{s \in \mathcal{K}} |f(s)| \leq \frac{\sqrt{\varepsilon/\pi}}{d}.$$

Proof. The lemma can be found in Gonek [9] (Lemma 2.5). □

Lemma 5. *Let a_1, \dots, a_N be real numbers linearly independent over the rational numbers. Let γ be a region of the N -dimensional unit cube with volume V (in*

the Jordan sense). Let $I_\gamma(T)$ be the sum of the intervals between $t = 0$ and $t = T$ for which the point $(a_1 t, \dots, a_N t) \pmod 1$ is inside γ . Then

$$\lim_{T \rightarrow \infty} \frac{I_\gamma(T)}{T} = V.$$

Proof. This is Theorem 1 in Appendix, Section 8, of Voronin and Karatsuba [11]. \square

For a curve $\omega(t)$ in \mathbb{R}^N we introduce the notation

$$\{\omega(t)\} = (\omega_1(t) - [\omega_1(t)], \dots, \omega_N(t) - [\omega_N(t)]),$$

where $[x]$ denotes the integral part of $x \in \mathbb{R}$.

Lemma 6. *Suppose that the curve $\omega(t)$ is uniformly distributed $\pmod 1$ in \mathbb{R}^N . Let D be a closed and Jordan measurable subregion of the unit cube in \mathbb{R}^N and let Ω be a family of complex-valued continuous functions defined on D . If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\{\omega(t)\}) 1_D(t) dt = \int_D f(x_1, \dots, x_N) dx_1 \dots dx_N$$

uniformly with respect to $f \in \Omega$, where $1_D(t)$ is equal to 1 if $\omega(t) \in D \pmod 1$, and 0 otherwise.

Proof. The lemma is Theorem 3 in Appendix, Section 8, of Voronin and Karatsuba [11]. \square

Lemma 7. *Let p_n be the n th prime number and $1 = d_1, d_2, \dots, d_l$ be algebraic real numbers which are linearly independent over \mathbb{Q} . Then the set $\{d_k \log p_n\}_{n \in \mathbb{N}}^{1 \leq k \leq l}$ is linearly independent over \mathbb{Q} .*

Proof. This is Proposition 2.2 in Nakamura [15]. The proof is based on Baker's [4, Theorem 2.4] result. \square

3. PROOF OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. We define a truncated Dirichlet L -function

$$L_v(s, \chi) = \prod_{p \leq v} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Roughly speaking, we first prove Theorem 1 for the truncated Dirichlet L -function and later we show that the tail is small.

Let $\{d_1, d_2, \dots, d_l\}$ be a maximal linearly independent (over \mathbb{Q}) subset of the set $\{d_1, d_2, \dots, d_m\}$. Then there are integers $a \neq 0$ and $a_{k,1}, a_{k,2}, \dots, a_{k,l}$ such that

$$(5) \quad d_k = \frac{1}{a} (a_{k,1}d_1 + a_{k,2}d_2 + \dots + a_{k,l}d_l) \quad \text{for } l < k \leq m.$$

Let

$$A = \max_{l < k \leq m} \{|a_{k,1}| + |a_{k,2}| + \cdots + |a_{k,l}|\}.$$

Denote by $\|x\|$ the minimal distance of $x \in \mathbb{R}$ to an integer. If

$$(6) \quad \left\| \tau \frac{d_n \log p}{2\pi a} \right\| < \delta \quad \text{for } p \leq v \text{ and } 1 \leq n \leq l$$

then

$$\left\| \tau \frac{d_n \log p}{2\pi} \right\| < a\delta \quad \text{for } p \leq v \text{ and } 1 \leq n \leq l$$

and, by the relation (5),

$$\left\| \tau \frac{d_k \log p}{2\pi} \right\| < A\delta \quad \text{for } p \leq v \text{ and } l < k \leq m.$$

By this and by the continuity of the function $L_v(s, \chi)$ we have that for any $\varepsilon > 0$ there is $\delta > 0$ such that for τ satisfying (6)

$$(7) \quad \max_{1 \leq k, n \leq m} \max_{s \in \mathcal{K}} |\log L_v(s + id_k \tau, \chi) - \log L_v(s + id_n \tau, \chi)| < \varepsilon.$$

For positive numbers δ , v , and T we define the set

$$(8) \quad S_T = S_T(\delta, v) = \left\{ \tau : \tau \in [0, T], \left\| \tau \frac{d_n \log p}{2\pi a} \right\| < \delta, p \leq v, 1 \leq n \leq l \right\}.$$

Let U be an open bounded rectangle with vertices on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$, such that the set \mathcal{K} is in U . Let $y > v$. We have

$$\begin{aligned} & \frac{1}{T} \int_{S_T} \int_U \sum_{k=1}^m |\log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi)|^2 d\sigma dt d\tau \\ &= \sum_{k=1}^m \int_U \frac{1}{T} \int_{S_T} |\log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi)|^2 d\tau d\sigma dt. \end{aligned}$$

For the inner integrals of the right-hand side of the last equality we will apply Lemma 6. Let p_n be the n th prime number. There are indexes M and N such that $p_M \leq v < p_{M+1}$ and $p_N \leq y < p_{N+1}$. By generalized Kronecker's theorem (Lemma 5) and by Lemma 7 the curve

$$\omega(\tau) = \left(\tau \frac{d_k \log p_n}{2\pi a} \right)_{1 \leq n \leq N}^{1 \leq k \leq l}$$

is uniformly distributed mod 1 in \mathbb{R}^{lN} . Let R' be a subregion of the lN -dimensional unit cube defined by inequalities

$$\|y_{k,n}\| \leq \delta \quad \text{for } 1 \leq k \leq l \text{ and } 1 \leq n \leq M$$

and

$$\left| y_{k,n} - \frac{1}{2} \right| \leq \frac{1}{2} \quad \text{for } 1 \leq k \leq l \text{ and } M+1 \leq n \leq N.$$

Let R be a subregion of the lM -dimensional unit cube defined by inequalities

$$\|y_{k,n}\| \leq \delta \quad \text{for } 1 \leq k \leq l \text{ and } 1 \leq n \leq M.$$

Clearly

$$\text{meas } R' = \text{meas } R = (2\delta)^{lM}.$$

Note that

$$\begin{aligned} & \log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi) = \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \\ (9) \quad & = - \sum_{v < p \leq y} \log \left(1 - \frac{\chi(p)}{p^{s+id_k \tau}} \right) = \sum_{v < p \leq y} \sum_{j=1}^{\infty} \frac{\chi^j(p)}{j p^{j(s+id_k \tau)}} \\ & = \sum_{M < n \leq N} \sum_{j=1}^{\infty} \frac{\chi^j(p_n)}{j p_n^{j(s+id_k \tau)}}. \end{aligned}$$

Thus in view of the linear dependence (5) we get

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \int_{S_T} \sum_{k=1}^m \left| \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \right|^2 d\tau \\ & = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{S_T} \left(\sum_{k=1}^l \left| \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \right|^2 \right. \\ & \quad \left. + \sum_{k=l+1}^m \left| \log \frac{L_y}{L_v} \left(s + \frac{i}{a} (a_{k,1}d_1 + a_{k,2}d_2 + \cdots + a_{k,l}d_l) \tau, \chi \right) \right|^2 \right) d\tau. \end{aligned}$$

By Lemma 6 and equality (9) we obtain that the last limit is equal to

$$\begin{aligned}
& \int_{R'} \left(\sum_{k=1}^l \left| \sum_{M < n \leq N} \sum_{j=1}^{\infty} \frac{\chi^j(p) e^{-2\pi i j a y_{k,n}}}{j p_n^{j\sigma}} \right|^2 \right. \\
& \quad \left. + \sum_{k=l+1}^m \left| \sum_{M < n \leq N} \sum_{j=1}^{\infty} \frac{\chi^j(p) e^{2\pi i j (a_{k,1} y_{1,n} + a_{k,2} y_{2,n} + \dots + a_{k,l} y_{l,n})}}{j p_n^{j\sigma}} \right|^2 \right) dy_{1,1} \dots dy_{l,N} \\
& = \text{meas } R \int_0^1 \dots \int_0^1 \left(\sum_{k=1}^l \left| \sum_{M < n \leq N} \sum_{j=1}^{\infty} \frac{\chi^j(p) e^{-2\pi i j a y_{k,n}}}{j p_n^{j\sigma}} \right|^2 \right. \\
& \quad \left. + \sum_{k=l+1}^m \left| \sum_{M < n \leq N} \sum_{j=1}^{\infty} \frac{\chi^j(p) e^{2\pi i j (a_{k,1} y_{1,n} + a_{k,2} y_{2,n} + \dots + a_{k,l} y_{l,n})}}{j p_n^{j\sigma}} \right|^2 \right) dy_{1,M+1} \dots dy_{l,N} \\
& = m \text{meas } R \sum_{v < p \leq y} \sum_{j=1}^{\infty} \frac{1}{j p^{2j\sigma}} \ll \text{meas } R \sum_{p > v} \frac{1}{p^{2\sigma}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
(10) \quad & \frac{1}{T} \int_{S_T} \int_U \sum_{k=1}^m |\log L_y(s + id_k \tau, \chi) - \log L_v(s + id_k \tau, \chi)|^2 d\sigma dt d\tau \\
& \ll \text{meas } R \sum_{p > v} \frac{1}{p^{2\sigma_1}}.
\end{aligned}$$

Again, by generalized Kronecker's theorem (Lemma 5),

$$(11) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \text{meas } S_T = \text{meas } R.$$

By (10) and (11), for large v , as $T \rightarrow \infty$, we have

$$\text{meas} \left\{ \tau : \tau \in S_T, \int_U \sum_{k=1}^m \left| \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \right|^2 d\sigma dt < \sqrt{\sum_{p > v} \frac{1}{p^{2\sigma_1}}} \right\} > \frac{1}{2} T \text{meas } R.$$

Then Lemma 4 gives

$$\begin{aligned}
& \text{meas} \left\{ \tau : \tau \in S_T, \max_{s \in \mathcal{K}} \sum_{k=1}^m \left| \log \frac{L_y}{L_v}(s + id_k \tau, \chi) \right|^2 d\tau \leq \frac{1}{d\sqrt{\pi}} \left(\sum_{p > v} \frac{1}{p^{2\sigma_1}} \right)^{\frac{1}{4}} \right\} \\
& > \frac{1}{2} T \text{meas } R,
\end{aligned}$$

where $d = \min_{z \in \partial U} \min_{s \in \mathcal{K}} |s - z|$. By the continuity of the logarithm we obtain that for any $\varepsilon > 0$ there is $v = v(\varepsilon)$ such that for any $y > v$

$$(12) \quad \text{meas} \left\{ \tau : \tau \in S_T, \max_{s \in \mathcal{K}} \sum_{k=1}^m |L_y(s + id_k \tau, \chi) - L_v(s + id_k \tau, \chi)|^2 d\tau < \varepsilon \right\} \\ > \frac{1}{2} T \text{ meas } R.$$

Now we will prove that for any $\delta > 0$ there is $y = y(\delta)$ such that

$$(13) \quad \text{meas} \left\{ \tau : \tau \in [0, T], \max_{s \in \mathcal{K}} \sum_{k=1}^m |L(s + id_k \tau, \chi) - L_y(s + id_k \tau, \chi)|^2 d\tau < \delta \right\} \\ > (1 - \delta) T.$$

The last formula together with (7), (8) and (12) yields Theorem 1. We return to the proof of (13). By the mean value theorem of the Dirichlet L -function (Steuding [17], Corollary 6.11) and by Carlson's Theorem (Titchmarsh [18], Chapter 9.51) we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |L(s + ix\tau, \chi) - L_y(s + ix\tau, \chi)|^2 d\tau = \sum_{n > y} \frac{|\chi(n)|}{n^{2\sigma}},$$

where x is fixed. Thus (13) follows in view of

$$\int_0^T \int_U \sum_{k=1}^m |L(s + id_k \tau, \chi) - L_y(s + id_k \tau, \chi)|^2 d\sigma d\tau \ll \sum_{n > y} \frac{|\chi(n)|}{n^{2\sigma_1}}.$$

Theorem 1 is proved. □

The proof of Theorem 2 is based on the ideas of Mauclaire [13], [14]. It uses the theory of Besicovitch almost periodic functions. We recall related definitions.

Let

$$P(\tau) = \sum_{n \in F} a_n e^{i\lambda_n \tau},$$

where F is a finite set, λ_n are any real numbers, and the coefficients a_n are any complex numbers. For real τ we say that $P(\tau)$ is a *trigonometric polynomial*.

A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is called *uniformly almost periodic (U.A.P.)* if given any $\varepsilon > 0$, there exists a trigonometric polynomial $P(\tau)$ such that

$$\sup_{\tau \in \mathbb{R}} |f(\tau) - P(\tau)| \leq \varepsilon.$$

A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is called B^q almost periodic ($B^q.A.P.$), $q \geq 1$, if given any $\varepsilon > 0$, there exists a trigonometric polynomial $P(\tau)$ such that

$$(14) \quad \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\tau) - P(\tau)|^q d\tau \leq \varepsilon.$$

If $q = 1$ then we write $B.A.P.$ (Besikovitch almost periodic) instead of $B^1.A.P.$. For any $q \geq 1$ it is clear that every $U.A.P.$ function is $B^q.A.P.$ and that every $B^q.A.P.$ function is $B.A.P.$

Proof of Theorem 2. Let

$$g(\tau) = \max_{1 \leq j, k \leq m} \max_{s \in \mathcal{K}} |L(s + id_j \tau, \chi_j) - L(s + id_k \tau, \chi_k)|$$

and let

$$F_T(x) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : g(\tau) < x \}$$

be a distribution function of $g(\tau)$. If $g(\tau)$ is $B.A.P.$ then it is known (see Jessen and Wintner [7, Theorem 27] or Laurinćikas [12, Theorem 6.3, Chapter 2]) that there is a distribution function $F(x)$ such that $F_T(x)$ converges weakly to $F(x)$ for $T \rightarrow \infty$. It means that if $F(x)$ is continuous at $x = \varepsilon$ then

$$\lim_{T \rightarrow \infty} F_T(\varepsilon)$$

exists. Thus to obtain Theorem 2 we need to show that $g(\tau)$ is $B.A.P.$

We remark that if $a(t)$ and $b(t)$ are both non-negative $B.A.P.$ functions of t , then, $t \mapsto \max(a(t), b(t))$ is also $B.A.P.$ since $\max(a(t), b(t))$ can be written as

$$\max(a(t), b(t)) = \frac{1}{2} (|a(t) - b(t)| + (a(t) + b(t))),$$

and the modulus of $B.A.P.$ function is again $B.A.P.$ By this we have that $g(\tau)$ is $B.A.P.$ if the function

$$f(\tau) = \max_{s \in \mathcal{K}} |L(s + id_1 \tau, \chi_1) - L(s + id_2 \tau, \chi_2)|$$

is $B.A.P.$ In view of the note below the formula (14) the function $f(\tau)$ is $B.A.P.$ if there are $U.A.P.$ functions $f_N(\tau)$ such that

$$(15) \quad \lim_{N \rightarrow +\infty} \left(\limsup_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T |f(\tau) - f_N(\tau)|^2 d\tau \right) = 0.$$

Let

$$L_N(s, \chi) = \sum_{n \leq N} \frac{\chi(n)}{n^s}$$

be a partial sum of the Dirichlet series associated with $L(s, \chi)$. Next we show that the equality (15) is true with

$$f_N(\tau) = \max_{s \in \mathcal{K}} |L_N(s + id_1 \tau, \chi_1) - L_N(s + id_2 \tau, \chi_2)|.$$

By repeating the proof of Proposition 12 of Maucilaire [13] we get that $f_N(\tau)$ is *U.A.P.* for any $d_1, d_2 \in \mathbb{R}$. Note that the case when d_1 or d_2 is equal to zero is already included in Proposition 12 of Maucilaire [13].

Further we have that

$$\begin{aligned} & L(s + id_1\tau, \chi_1) - L(s + id_2\tau, \chi_2) \\ &= (L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1) + L_N(s + id_2\tau, \chi_2) - L(s + id_2\tau, \chi_2)) \\ & \quad + (L_N(s + id_1\tau, \chi_1) - L_N(s + id_2\tau, \chi_2)), \end{aligned}$$

and as a consequence, we get that

$$\begin{aligned} & |f(\tau) - f_N(\tau)| \\ & \leq \sup_{s \in K} |L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1) + L_N(s + id_2\tau, \chi_2) - L(s + id_2\tau, \chi_2)| \\ & \leq \sup_{s \in K} |L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1)| \\ & \quad + \sup_{s \in K} |L_N(s + id_2\tau, \chi_2) - L(s + id_2\tau, \chi_2)|. \end{aligned}$$

Then, in view of the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, we obtain that

$$\begin{aligned} & \frac{1}{2T} \int_{-T}^T |f(\tau) - f_N(\tau)|^2 dt \\ & \leq \frac{1}{T} \int_{-T}^T \left(\sup_{s \in K} |L(s + id_1\tau, \chi_1) - L_N(s + id_1\tau, \chi_1)| \right)^2 dt \\ & \quad + \frac{1}{T} \int_{-T}^T \left(\sup_{s \in K} |L_N(s + id_2\tau, \chi_2) - L(s + id_2\tau, \chi_2)| \right)^2 dt. \end{aligned}$$

By Maucilaire [14, Theorem 5.1] we have that, for any real d ,

$$\lim_{N \rightarrow +\infty} \left(\lim_{T \rightarrow +\infty} \sup_{T'} \frac{1}{2T'} \int_{-T}^T \left(\sup_{s \in K} |f(s + idt) - f_N(s + idt)| \right)^2 dt \right) = 0.$$

This proves the equality (15) and Theorem 2 □

From the proof we see that Theorem 2 remains true with Dirichlet L -functions $L(s, \chi_j)$, $j = 1, \dots, m$, replaced by any general Dirichlet series satisfying conditions of Theorem 5.1 of Maucilaire [14].

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Remark. The ‘lim inf’ version of Corollary 3 is independently obtained by Takashi Nakamura in “The generalized strong recurrence for non-zero rational parameters”, arXiv:1006.1778v1 [math.NT].

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